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THE STOKES' PROBLEM FOR THE ELLIPSOID USING ELLIPSOIDAL KERNELS

by

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Foreword

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Acknowledgements

Abstract

A brief review of Stokes' problem for the ellipsoid as a reference surface is given. Another solution of the problem using an ellipsoidal kernel, which represents an iterative form of Stokes' integral, is suggested with a relative error of the order of the flattening. On studying of Rapp's method in detail the procedures of improving its convergence are discussed.

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1. Introduction

The determination of the shape of the global earth is
one of the topics pursued in physical geodesy, while the inves-
tigation of the earth's shape consists essentially in eval-
uating the height anomaly and the deflection of the vertical
by using the gravity anomaly information at every point on
the surface of the earth. Since 1849 Stokes developed a formula,
which states the disturbing potential of the earth can be
derived from gravity anomaly values at the geoid, his famous
formula has been traditionally regarded as a basis to determine
the earth's figure and extensively applied to practice. But
there are two important limitations of its practical appli-
cation in geodesy. That is to say this formula is valid only
for (a) spherical boundary surface, and (b) distribution
of gravity values over the geoid instead of the topographical
surface. With spherical approximation it implied that the
error of the order of f , the flattening of the earth, will be
certainly introduced into the computed results. For instance,
the absolute effect of this relative error on the geoid undulation
N reached $fN \approx 0.7$ meters when the geoid undulation is 100
meters (Fang Tsün, 1973, p.236). Although this error is usually
permissible for normal purposes, it should be taken into consid-
eration for special investigation like the accurate determination
of the figure and gravity field of the earth. In addition,
the requirement that the gravity values must refere to the
geoid and no masses can lie outside the geoid is so stern
that it is impossible for this requirement to be exactly satisfied
even by using various calculation procedures to reduce gravity
and regularize the surface of the earth's geoid. Therefore,
from both theoretical and practical points of view, we cannot
use the classical equation of Stokes for analysing in detail
the earth's geoid; instead we have to develop modern theories
and techniques. During the last 40 years or so there has
been a great amount of work being done in this field aiming
to improve and develop the classical Stokes' theory. It should
be pointed out that the most important contributions have
been made by Molodensky (1962), Zagrebin (1956, 1965), and
Pellinen (1981). They approached the problem in two different
ways. Molodensky gave up the traditional concept of the geoid
of the earth and, instead, introduced an auxiliary geometric
surface, the so-called telluroid, to reduce the problem to
the solution of Laplace boundary value problem of mixed type
on the telluroid, which is called the geodetic boundary value
problem in geodesy. Thus, we can find the height anomaly
and the deflection of the vertical directly by using gravity
anomaly data measured on the actual surface of the earth.
Zagrebin sought a solution of the boundary value problem with
the ellipsoid as a reference surface to carry out the investi-
gation of the gravity field of the regularized earth. Both
these theories establish the most elaborate form of the so-
called model approach to physical geodesy.

In the present report, we are not going to discuss Molodensky's problem, which can be found in the authoritative work ADVANCED PHYSICAL GEODESY (Moritz, 1980); instead we are just planning to try to given a brief review of the state of art about the Zagrebin's problem, the problem of Stokes for the ellipsoid. Further-more, we will place stress on the new techniques which are appropriate for practical application, such as those developed by Rapp (1981). In addition, we also contribute another solution with iterative form for the problem of Zagrebin using an ellipsoidal kernel, which is similar to Molodensky's (1962, p.53) and Koch's (1968, p.22). Moreover, on studying Rapp's method in detail, we put forward some new designs for improving the convergence of his method.

2. The Form of the Solution

Since the Zagrebin's solution was published a large number of authors have been devoted to solving the Stokes' problem for a reference ellipsoid with a relative error of the order of the flattening. Looking into all works it should be found that allavailable approaches, from the mathematical point of view, could be classified into two groups: the direct solution - series approach, expanding the anomalous potential of the earth into a series of harmonics, such as Zagrebin's (1956, 1965) and Bjerhammar's (1962), and the indirect solution - ellipsoidal kernel, reducing this problem to the solution of the Fredholm's integral equation of the 2nd kind by means of either the third Green's formula or the theory of the potential of surface layer, such as Molodensky's (1962), Fang Tsün's (1965), Koch's (1968) and Lelgemann's (1970), etc.

In spite of using any methods in these two groups to solve the problem of Stokes for an ellipsoid of revolution the solution finally obtained could be written as the following form:

(1) $T_E = T + \delta T$

or from the Bruns' formula,

(2) $N_E = N + \delta N$

where T or N is the principal part of the solution, expressed by Stokes' equation, which characterizes the fundamental properties of the gravity field of the earth, and δT or δN is a correction term, which responds to the compression of the earth.

In case of the direct approaches, the solutions are represented as an open form, the series of harmonics; otherwise, a closed form, the Stokes' integral.

We can see that the series treatments are not only tedious in derivation, but also very complicated in final form obtained. For example, for the Zagrebin's solution it is necessary to carry out not less than nine integrals over the reference surface in order to obtain N_E of one point. Thus it can be seen that such a kind of solution is not appropriate for practical application. In view of this situation, another way, using an ellipsoidal kernel, to approach the problem was suggested by Molodensky, and afterwards developed by Fang Tsün, and Koch, etc. From the pure mathematical point of view, the solution of ellipsoidal kernel methods, which is expressed as a set of iterative integral forms, is graceful and strict, and convenient for theoretical analysis. However, unfortunately, it should be troublesome to employ them to evaluate geoid undulations through potential coefficient information. For this reason, it is necessary to improve and develop the kernel methods so as to suit them to the need of the modern practice in geodesy, Before discussing this point, we are going to contribute another solution of the Zagrebin's problem with the aid of ellipsoidal kernel in the following section.

3. Another Solution of the Problem Using an Ellipsoidal Kernel

Most authors utilize the third Green's theorem to seek the solution of the Zagrebin's problem; instead we still start with the theory of the potential of a simple layer as Koch (1968) did, but a special kind of coordinate transformation is employed, by which the whole procedures of derivation is significantly simplified. Because of simplification, the approach developed here should be meaningful. Now, we will show the derivation as follows.

Assume the equation of the reference surface is given by

(3) $\sigma : \dfrac{x^2 + y^2}{a^2} + \dfrac{z^2}{c^2} = 1$

First of all, let us introduce the coordinate transformation

$A \big|_{(x,y,z) \rightarrow (\mu,\beta,\lambda)}:$

(4) $x = E \cosh\mu \cos\beta \cos\lambda$

       $y = E \cosh\mu \cos\beta \sin\lambda$

       $z = E \sinh\mu \sin\beta$

       $E = (a^2 - c^2)^{\frac{1}{2}} = c\,e'^2$

Obviously, the ellipsoidal coordinate system (μ,β,λ) is orthogonal; and β and λ are nothing but the so-called reduced latitude and longitude respectively. We then have an element of surface:

(5) $\qquad d\sigma = G_\beta\ G_\lambda\ d\beta\ d\lambda = G_\beta\ \bar{G}_\lambda\ ds$

an element of outer normal to σ :

(6) $\qquad dn = G_\mu\ d\mu$

where G_μ , G_β and G_λ are called the first Gaussian fundamental quantities defined by

(7) $\qquad G_\mu = [(\frac{\partial x}{\partial \mu})^2 + (\frac{\partial y}{\partial \mu})^2 + (\frac{\partial z}{\partial \mu})^2]^{\frac{1}{2}} = E[\cosh^2\mu - \cos^2\beta]^{\frac{1}{2}}$

$\qquad\qquad G_\beta = [(\frac{\partial x}{\partial \beta})^2 + (\frac{\partial y}{\partial \beta})^2 + (\frac{\partial z}{\partial \beta})^2]^{\frac{1}{2}} = E[\cosh^2\mu - \cos^2\beta]^{\frac{1}{2}}$

$\qquad\qquad G_\lambda = [(\frac{\partial x}{\partial \lambda})^2 + (\frac{\partial y}{\partial \lambda})^2 + (\frac{\partial z}{\partial \lambda})^2]^{\frac{1}{2}} = E\ \cosh\mu\cos\beta = \bar{G}_\lambda\cos\beta$

It should be very easy to get their values on σ :

(10) $\qquad G_\mu = G_\beta = c(1 + e'^2\ \sin\beta)^{\frac{1}{2}} + \alpha(e'^2)$

(11) $\qquad \bar{G}_\lambda = E\ \cosh\mu = a$

Secondly, let us consider the Zagrebin's problem in the space (μ,β,λ) rather than (x,y,z). As a starting point we will take the potential T of the simple layer on σ . With the action of the operator A the potential T , the outer limit value of the derivative of T in the direction of the normal on σ , and the boundary condition which the potential must satisfy becomes respectively:

(12) $\qquad T_P = \iint\limits_{s} \frac{\phi}{r}\ G_\beta\ \bar{G}_\lambda\ ds \qquad\qquad P\epsilon C\upsilon$

(13) $\qquad \frac{\partial T}{\partial \mu}_P = \iint\limits_{s} \phi\frac{\partial}{\partial \mu_P}(\frac{1}{r})\ G_\beta\ \bar{G}_\lambda\ ds - 2\pi\ \phi_P\ G_{\mu P} \qquad P\epsilon\sigma$

(14) $\qquad \frac{\partial T}{\partial \mu}_P - \frac{T}{\gamma_P}\frac{\partial \gamma}{\partial \mu}_P = -\Delta g\ G_\mu \qquad\qquad P\epsilon\sigma$

where s is a unit sphere, r , the distance from the parameter point P to the integral variable point Q , and ϕ means the density distribution function on the surface, $C\upsilon$, the space outside the ellipsoid.

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Combining (12), (13) with (14), it follows that the Fredholm integral equation of the 2nd kind with respect to the function ϕ will be

$$(15) \quad 2\pi\phi = \Delta g + \iint_S \phi K(p,Q)ds$$

where

$$(16) \quad K(P,Q) = [\frac{\partial}{\partial \nu_p}(\frac{1}{r}) - \frac{1}{r\gamma_p}\frac{\partial\gamma}{\partial\nu_p}] \; G_\mu^{-1} \; G_\beta \; \bar{G}_\lambda$$

is the kernel of the integral equation. In our circumstance, it is also called the ellipsoidal kernel, because it is produced from the boundary surface of the ellipsoid. All quantities without the subscript P in (16) refer to the integral variable point Q, but those outside the integral sign in (15) are always taken at the parameter point P on σ.

In order to solve the equation (15), it is necessary to represent the kernel (16) in more detail. The whole procedure is described as follows.

It is clear that we have

$$(17) \quad \begin{cases} \vec{r} = \vec{\rho} - \vec{\rho}_p = (x - x_p)\,\vec{i} + (y - y_p)\,\vec{j} + (z - z_p)\,\vec{k} \\ r^2 = 4\rho\rho_p \sin^2\frac{\psi}{2} + (\rho - \rho_p)^2 \end{cases}$$

and

$$(18) \quad \rho = (x^2 + y^2 + z^2)^{\frac{1}{2}} = E(\sinh^2\mu + \cos^2\beta)^{\frac{1}{2}} = c(1 + e'^2\cos^2\beta)$$

where ψ is the angle between the vector radii $\vec{\rho}$ and $\vec{\rho}_p$.

Since the quantity $\rho - \rho_p$ is equal to $O(e'^2)$:

$$(19) \quad \rho - \rho_p = \tfrac{1}{2} e'^2(\cos^2\beta - \cos^2\beta_p)$$

the quantity $(\rho - \rho_p)^2$ could be ignored, and (17) then becomes

$$(20) \quad r^2 = 4\rho\rho_p \sin^2\frac{\psi}{2}$$

Substituting (18) into (20), we arrive at

$$(21) \quad r^2 = 4c^2\sin^2\frac{\psi}{2} \{1 + \tfrac{1}{2} e'^2(\cos^2\beta + \cos^2\beta_p)\}$$

(22) $\quad \dfrac{1}{r} = \dfrac{1}{2c \sin\frac{\psi}{2}} \{1 - \frac{1}{4} e'^2 (\cos^2\beta + \cos^2\beta_p)\}$

On the other hand, from (17) we have

$$\frac{\partial r}{\partial \mu_p} = -\frac{\tanh\mu_p}{2r} \{r^2 + \rho_p^2 - \rho^2 - \frac{2(z - z_p)z_p}{\sinh^2\mu_p}\}$$

Using (21), (18), (4-3) and (4-4) it is not difficult to bring this derivative to the following form:

(23) $\quad \dfrac{\partial r}{\partial \mu_p} = \dfrac{r\tanh\mu_p}{2} \{1 + \dfrac{e'^2}{4\sin^2\frac{\psi}{2}} (\sin\beta_p - \sin\beta)^2\}$

Thus, we finally get

(24) $\quad \dfrac{\partial}{\partial \mu_p}(\dfrac{1}{r}) = -\dfrac{1}{r^2}\dfrac{\partial r}{\partial \mu_p} = -\dfrac{G_{\mu p}}{4ac\sin\frac{\psi}{2}}\{(1 - \frac{1}{4}e'^2\sin^2\beta_p -$

$\qquad\qquad - \frac{1}{4}e'^2(\cos^2\beta_p + \cos^2\beta) + \dfrac{e'^2}{4\sin^2\frac{\psi}{2}}(\sin\beta_p - \sin\beta)^2\}$

We shall next derive the expression for the second term in the bracket of (16).

The Bruns' formula in our coordinate system (μ,β,λ) should be the following form

(25) $\quad \dfrac{1}{\gamma_p}\dfrac{\partial\gamma}{\partial\mu_p} = -G_{\mu p}\{(\dfrac{1}{N_p} + \dfrac{1}{M_p}) - \dfrac{2\omega^2}{\gamma_p}\}$

By means of the formulas of the radii of curvature in the meridian and in the prime vertical

(26) $\quad N = a(1 + e'^2\sin^2\beta)^{\frac{1}{2}}$

(27) $\quad M \approx \dfrac{c^2}{a}(1 + \dfrac{3}{2} e'^2\sin^2\beta)$

we immediately obtain

(28) $\quad \dfrac{1}{\gamma_p}\dfrac{\partial\gamma}{\partial\mu_p} = -\dfrac{2}{a} G_{\mu p}\{1 + \frac{1}{2}e'^2 - e'^2\sin^2\beta_p + q\}$

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where

(29) $\quad q = \dfrac{a \, \omega^2}{\gamma_p}$

Hence, we have

(30) $\quad \dfrac{1}{r\gamma_p} \dfrac{\partial \gamma}{\partial \mu_p} = -\dfrac{G\mu_p}{ac \sin \frac{\psi}{2}} \{1 - \frac{3}{4} e'^2 \sin^2\beta_p + \frac{1}{4} e'^2 \sin^2\beta + q\}$

Consequently, with (10), (11), (24) and (30) we find the expression of the kernal (16) as follows

(31) $\quad K(P,Q) = K_0(P,Q) + K_1(P,Q)$

where

(32) $\quad K_0(P,Q) = \dfrac{3}{4 \sin\frac{\psi}{2}}$

(33) $\quad K_1(P,Q) = e'^2 K_0 \{\frac{1}{6} - \frac{11}{12} \sin^2\beta_p + \frac{9}{12} \sin^2\beta + \dfrac{1}{12 \sin^2 \frac{\psi}{2}} (\sin\beta_p -$

$$- \sin\beta)^2 + \frac{4}{3} q \}$$

Moreover, with (22), (10) and (11), (12) may be written as

(34) $\quad T = \dfrac{2}{3} a \displaystyle\iint_S \phi(K_0 + k) \, ds$

where

(35) $\quad k = e'^2 K_0 \{\frac{1}{2} - \frac{1}{4} (\cos^2\beta - \cos^2\beta_p)\}$

We are going to determine the disturbing potential T. For this purpose, we first solve the integral equation (15).

For convenience, (15) is rewritten as the operator form

(36) $\quad \phi = B\phi + \dfrac{G_0}{2\pi}$

where

(37) $\quad G_0 = \Delta g$

and the operator B is defined by

$$(38) \qquad B(\cdot) = \frac{1}{2\pi} \iint_S K(\cdot)\, ds$$

Splitting (36) in the order of magnitude of e'^2 without regard for $o(e'^2)$ we arrive at the following equations

$$(39) \qquad O(1): \quad \phi_0 = B_0\, \phi_0 + \frac{G_0}{2\pi}$$

$$(40) \qquad O(e'^2): \phi_1 = B_0\, \phi_1 + \frac{G_1}{2\pi}$$

where

$$(40') \qquad\qquad G_1 = B_1\, \phi_0$$

and

$$(41) \qquad B_1: B_1(\cdot) = \frac{1}{2\pi} \iint_S K_1(\cdot)\, ds$$

is called the i-th order operator with respect to e'^2.

It should be easy to verify the following lemma:

For any function X and G with the same order of e'^2, the operator equation

$$(42) \qquad X = B_0\, X + \frac{G}{2\pi}$$

always has a unique solution

$$(43) \qquad 2\pi X = \frac{3}{2a}\, \bar{S}G + G$$

where $\bar{S}$ is the Stokes operator:

$$(44) \qquad \bar{S}(\cdot) = \frac{a}{4\pi} \iint_S [S(\psi) - 1]\, (\cdot)\, ds$$

Applying this lemma to both equations (39) and (40) we have

$$(45) \qquad 2\pi\phi_0 = \frac{3}{2a}\, \bar{S}\, G_0 + G_0$$

and

(46) $\quad 2\pi\phi_1 = \dfrac{3}{2a}\,\overline{S}\,G_1 + G_1$

    Consequently, we get the solution of the equation (36) as follows

(47) $\quad \phi = \phi_0 + \phi_1$

    Finally, we are going to determine potential T.

    The expression (34) may be rewritten as

(48) $\quad T = \dfrac{2}{3}\,a[2\pi B_0\phi + \dfrac{3}{2a}\,G_{11}]$

and also split into two parts in the order of e'^2:

(49) $O(1)$: $\quad T_0 = \dfrac{2}{3}\,a[2\pi B_0\phi_0]$

(50) $O(e'^2)$: $\quad T_1 = \dfrac{2}{3}\,a[2\pi B_0\phi_1 + \dfrac{3}{2a}\,G_{11}]$

(51) $\quad G_{11} = \dfrac{2}{3}a \iint\limits_{S} \phi_0 k\,d\,s$

    With (39), (45), (40) and (46) we get the solution of the Stokes' problem for the ellipsoid of rotation with a relative error of the flattening as follows

(52) $\quad T = T_0 + T_1$

with

(53) $\quad T_0 = \overline{S}\,G_0$

(54) $\quad T_1 = \overline{S}\,G_1 + G_{11}$

or

(55) $\quad T_0 = \dfrac{a}{4\pi} \iint\limits_{S} G_0[S(\psi)-1]\,d\,s\,, \quad G_0 = \Delta g$

$$(56) \quad T_1 = \frac{a}{4\pi} \iint_S G_1 [S(\psi) - 1] \, ds + \frac{2}{3} a \iint_S \phi_0 k \, ds, \quad G_1 = \frac{1}{2\pi} \iint_S K_1 \phi_0 \, ds$$

Clearly, if the reduced coordinate system (β, λ) is transformed into geocentric coordinate system (ϕ, λ), then our solution is reduced to the Koch's solution (Koch, 1968, p.22).

4. Truncation Method of Rapp

As mentioned above, the advantage of ellipsoidal kernel approaches consist in its closed iterative integral form. But in practical application, as we know, only numerical treatments can be carried out for these integrals. Moreover, they are extended over the global earth, so that the calculation is fairly tedious and laborious even in order to obtain the geoid undulation of one point at the earth's surface. It is needless to say that it is much more difficult to evaluate the geoid undulations of several points. Above all, a more important fact is that today people cannot have a uniform and dense distribution of gravity anomaly data on the surface of the earth. In this case, it is impossible to expect to arrive at good results through the procedures of global numerical integration. It follows that we have to find the other successful ways to reach our purpose. Even though the series methods are also complicated for practice, they bring light to the probability to use the information of potential coefficients to determine the geoid undulations. For containing the advantages of both series and ellipsoidal kernel approaches and overcoming their disadvantages a new ellipsoidal kernel procedure to compute the geoid undulation is developed by Lelgemann (1970). The principal part of his solution is exactly Stokes integral formula, and the correction term consists of a series of harmonic functions, unlike the general ellipsoidal kernel methods. Thus, we can use Lelgemann's formula to compute the ellipsoidal correction through the potential coefficient information. Later, a somewhat revised approach to the problem is suggested by Moritz (1980, p.320).

It should be mentioned a well-known fact that the convergence of series of harmonic functions may not be very good. Theoretically, we must use a large number of terms to expect a rather good computed result. But in fact, the number of terms in the series of harmonic functions is only finite, in spite of either using gravity data measured at the earth's surface or combining these with the information obtained by the aid of satellites. These coefficients cannot by any means represent the complicated situations of the gravity field of the earth. It follows that Lelgemann's solution only takes into account the long wavelength part of correction to the geoid undulation resulted from the earth's compression.

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Aiming at this defect, Rapp (1981) developed another new approach to calculate the ellipsoidal correction. He divided the solution of the problem into two parts: one is an integral over a cap whose center is the computation point, which is evaluated by using gravity data measured at the earth's surface, and another is a series, calculated by means of the potential coefficient information. Thus, not only the effect of flattening on the long wavelength global undulation is considered, but also on the local undulation. This approach is preferable up to now from both theoretical and practical points of view.

Rapp gives two kinds of formulas for the problem. We now write (Rapp, 1981):

The first kind (using regular Stokes' function):

(57) $\quad N_E = N_1 + N_2 + \Delta N_1 + \Delta N$

with

(58) $\quad N_1 = \frac{R}{4\pi\gamma} \int_0^{\psi_0} \int_0^{2\pi} \Delta g \, S(\psi) \, \sin\psi \, d\psi \, d\alpha$

(59) $\quad N_2 = \frac{R}{2\gamma} \sum_{n=2}^{\infty} Q_n \, \Delta g_n^{\circ}$

(60) $\quad \Delta N_1 = \frac{Re^2}{2\gamma} \sum_{n=0}^{\infty} (Q_n - \chi_n) \Delta g_n^1$

(61) $\quad \Delta N = e^2 (\frac{1}{4} - \frac{3}{4} \sin^2\phi) N$

where

(62) $\quad \Delta g^{\circ} = \sum_{n=2}^{\infty} \Delta g_n^{\circ} = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \frac{n-1}{R} (A_{nm} \cos m\lambda + B_{nm} \sin m\lambda) P_{nm}(\sin\phi)$

(63) $\quad \Delta g^1 = \sum_{n=0}^{\infty} \Delta g_n^1 = \frac{1}{R} \sum_{n=0}^{\infty} (G_{nm} \cos m\lambda + H_{nm} \sin m\lambda) P_{nm}(\sin\phi)$

(64) $\quad \chi_n = \begin{cases} 0 & , \; n < 2 \\ \frac{2}{n-1} & , \; n \geq 2 \end{cases}$

For the meanings of the other symbols see (Rapp, 1981).

As matter of fact, we can only calculate a finite sum of series; therefore we now give the truncation error of N_E :

$$(65) \quad \delta N_E = \delta N_1 + \delta N_2$$

where

$$(66) \quad \delta N_1 = \frac{Re^2}{2\gamma} \sum_{n=m+1}^{\infty} (Q_n - \chi_n)\Delta g_n^1$$

$$(67) \quad \delta N_2 = \frac{R}{2\gamma} \sum_{n=m+1}^{\infty} Q_n \Delta g_n^o$$

According to (Hsu and Zhu, 1979), we have the variance of the truncation error of geoid undulation as follows:

$$(68) \quad \sigma^2_{\delta N_E^I}(m,\psi_0) = \left(\frac{R}{2\gamma}\right)^2 \sum_{n=m+1}^{\infty} Q_n^2 \, \sigma_n^2 \, (\Delta g^o)$$

The second kind of computation (using the modified Stokes' function) is as follows:

$$(69) \quad N_E = N_1 + N_2 + \Delta N_1 + \Delta N$$

where

$$(70) \quad N_1 = \frac{R}{4\pi\gamma} \int_0^{\psi_0} \int_0^{2\pi} \Delta g[S(\psi) - S(\psi_0)] \sin\psi \, d\psi \, d\alpha$$

$$(71) \quad N_2 = \frac{R}{2\gamma} \sum_{n=2}^{\infty} \bar{Q}_n \, \Delta g_n^o$$

$$(72) \quad \Delta N_1 = \frac{Re^2}{2\gamma} \sum_{n=0}^{\infty} (\bar{Q}_n - \chi_n)\Delta g_n^1$$

$$(73) \quad \bar{Q}_n = Q_n(\psi_0) + \frac{S(\psi_0)}{n-1} [P_{n-1}(\cos\psi_0) - \cos\psi_0 \, P_n(\cos\psi_0)], \quad n > 1$$

The truncation error of N_E and its variance are expressed respectively as:

$$(74) \quad \delta N_E = \delta N_1 + \delta N_2$$

where

$$(75) \quad \delta N_1 = \frac{Re^2}{2\gamma} \sum_{n=m+1}^{\infty} (\bar{Q}_n - \chi_n)\Delta g_n^1$$

$$(76) \quad \delta N_2 = \frac{R}{2\gamma} \sum_{n=m+1}^{\infty} \bar{Q}_n \Delta g_n^o$$

and

$$(77) \quad \sigma^2_{\delta N_E^{II}}(m,\psi_0) = (\frac{R}{2\gamma})^2 \sum_{n=m+1}^{\infty} \bar{Q}_n^2 \, \sigma_n^2(\Delta g^\circ)$$

5. Further Improvement of Convergence of Rapp's Method

Rapp used the truncation theory to "remove" some local undulation terms from Stokes' series, the correction part in Lelgemann solution, into the integral over a cap surrounding the computation point, so that the convergence of the new series obtained is significantly accelerated, because there are not many local undulation terms in the series any more.

We now suggest two procedures for further improving the convergence of Rapp's method.

The first kind, using the optimum squares approximation to Stokes' function, is now described.

We introduce the following approximation function $S_m(\psi)$ to Stokes' function $S(\psi)$ at interval $[\psi_0,\pi]$:

$$(79) \quad S_m(\psi) = \sum_{n=0}^{m} \frac{2n+1}{2} K_n(m,\psi_0) \, P_n(\cos\psi)$$

where the coefficients $K_n(m,\psi_0)$ satisfy the following condition:

$$(80) \quad \int_{\psi_0}^{\pi} [S(\psi) - S_m(\psi)] \sin\psi d\psi = min$$

From (Fang Tsün, 1973, eq. (8.65)) we have:

$$(81) \quad K_n(m,\psi_0) = Q_n + \sum_{s=0}^{m} \frac{2s+1}{2} \chi_{ns} K_s(m,\psi_0)$$

$$(82) \quad \chi_{ns} = \int_0^{\psi_0} P_n(\cos\psi) P_s(\cos\psi) \sin\psi d\psi$$

Let us analytically extend the function $S_m(\psi)$ from region $[\psi_0,\pi]$ to the whole region $[0,\pi]$. Then, following (Rapp, 1981, eq. (40)), the solution of the problem can be expressed as

$$(83) \quad N_E = \frac{R}{4\pi\gamma} \int_0^{\psi_0} \int_0^{2\pi} (\Delta g - e^2 \Delta g^1) [S(\psi) - S_m(\psi)] \sin\psi d\psi d\alpha$$

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$$+ \frac{R}{4\pi\gamma} \int_0^\pi \int_0^{2\pi} (\Delta g - e^2 \Delta g^1) S_m(\psi) \sin\psi \, d\psi \, d\alpha$$

$$+ \frac{R}{4\pi\gamma} \int_{\psi_0}^\pi \int_0^{2\pi} (\Delta g - e^2 \Delta g^1) [S(\psi) - s_m(\psi)] \sin\psi \, d\psi \, d\alpha + \Delta N$$

We write (Rapp, 1981, eq. (23))

(84) $\Delta g = \Delta g^\circ + e^2 \Delta g^1$

From (79) and (84) the second integral in (83) can be written as:

(85) $N_2^1 = \dfrac{R}{4\pi\gamma} \int_0^\pi \int_0^{2\pi} (\Delta g - e^2 \Delta g^1) S_m(\psi) \sin\psi \, d\psi \, d\alpha =$

$$= \frac{R}{2\gamma} \sum_{n=2}^m K_n(m, \psi_0) \Delta g_n^\circ$$

We are going to calculate the integral involving $e^2 \Delta g^1$ in the first term of (84). We can write:

(86) $\dfrac{Re^2}{4\pi\gamma} \int_0^{\psi_0} \int_0^{2\pi} \Delta g^1 [S(\psi) - S_m(\psi)] \sin\psi \, d\psi \, d\alpha$

$$= \frac{Re^2}{4\pi\gamma} \int_0^\pi \int_0^{2\pi} \Delta g^1 [S(\psi) - S_m(\psi)] \sin\psi \, d\psi \, d\alpha -$$

$$- \frac{Re^2}{4\pi\gamma} \int_{\psi_0}^\pi \int_0^{2\pi} \Delta g^1 [S(\psi) - S_m(\psi)] \sin\psi \, d\psi \, d\alpha$$

We have:

(87) $\dfrac{Re^2}{4\pi\gamma} \int_0^\pi \int_0^{2\pi} \Delta g^1 [S(\psi) - S_m(\psi)] \sin\psi \, d\psi \, d\alpha$

$$= \frac{Re^2}{4\pi\gamma} \int_0^\pi \int_0^{2\pi} \Delta g^1 S(\psi) \sin\psi \, d\psi \, d\alpha -$$

$$- \frac{Re^2}{4\pi\gamma} \int_0^\pi \int_0^{2\pi} \Delta g^1 S_m(\psi) \sin\psi \, d\psi \, d\alpha$$

$$= \frac{Re^2}{2\gamma} \sum_{n=0}^\infty (\chi_n - Q_n) \Delta g_n^1 - \frac{Re^2}{2\gamma} \sum_{n=0}^m K_n(m, \psi_0) \Delta g_n^1$$

$$= \frac{Re^2}{2\gamma} \sum_{n=0}^m [\chi_n - Q_n - K_n(m, \psi_0)] \Delta g_n^1 +$$

$$+ \frac{Re^2}{2\gamma} \sum_{n=m+1}^\infty (\chi_n - Q_n) \Delta g_n^1$$

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From (Hsu and Zhu, 1979) we arrive at:

$$(88) \quad \frac{Re^2}{4\pi\gamma} \int_{\psi_0}^{\pi} \int_0^{2\pi} \Delta g^1 [S(\psi) - S_m(\psi)] \sin\psi \, d\psi \, d\alpha =$$

$$= \frac{Re^2}{2\gamma} \sum_{n=m+1}^{\infty} Q_n'(m,\psi_0) \Delta g_n^1$$

Hence, (86) becomes:

$$(89) \quad \frac{Re^2}{4\pi\gamma} \int_0^{\psi_0} \int_0^{2\pi} \Delta g^1 [S(\psi) - S_m(\psi)] \sin\psi \, d\psi \, d\alpha$$

$$= \frac{Re^2}{4\pi\gamma} \sum_{n=0}^{m} [\chi_n - Q_n - K_n(m,\psi_0)] \Delta g_n^1 +$$

$$+ \frac{Re^2}{2\gamma} \sum_{n=m+1}^{\infty} [\chi_n - Q_n - Q_n'(m,\psi_0)] \Delta g_n^1$$

$$(90) \quad Q_n' = \int_{\psi_0}^{\pi} [S(\psi) - S_m(\psi)] \, P_n(\cos\psi) \, \sin\psi \, d\psi$$

Following (Hsu and Zhu, ibid) and (84) the third integral in (83) can be written as:

$$(91) \quad \delta N_2' = \frac{R}{4\pi\gamma} \int_{\psi_0}^{\pi} \int_0^{2\pi} (\Delta g - e^2 \Delta g^1) [S(\psi) - S_m(\psi)] \sin\psi \, d\psi \, d\alpha$$

$$= \frac{R}{4\pi\gamma} \int_{\psi_0}^{\pi} \int_0^{2\pi} \Delta g^0 [S(\psi) - S_m(\psi)] \sin\psi \, d\psi \, d\alpha$$

$$= \frac{R}{2\gamma} \sum_{n=m+1}^{\infty} Q_n'(m,\psi_0) \Delta g_n^0$$

Therefore, the equation (83) finally becomes the following form:

$$(92) \quad N_E = N_1' + N_2' + \Delta N_1' + \delta N_E + \Delta N$$

where

$$(93) \quad N_1' = \frac{R}{4\pi\gamma} \int_0^{\psi_0} \int_0^{2\pi} \Delta g[S(\psi) - S_m(\psi)] \sin\psi \, d\psi \, d\alpha$$

$$(85) \quad N_2' = \frac{R}{2\gamma} \sum_{n=2}^{m} K_n(m,\psi_0) \Delta g_n^0$$

$$(94) \quad \Delta N_2' = \frac{Re^2}{2\gamma} \sum_{n=0}^{m} Y_n(m,\psi_0) \Delta g_n^1$$

(95) $\delta N_E' = \delta N_1' + \delta N_2'$

(96) $\delta N_1' = \frac{Re^2}{2\gamma} \sum_{n=m+1}^{\infty} Y_n'(m,\psi_0)\Delta g_n^1$

(91) $\delta N_2' = \frac{R}{2\gamma} \sum_{n=m+1}^{\infty} Q_n'(m,\psi_0)\Delta g_n^0$

(97) $Y_n(m,\psi_0) = K_n(m,\psi_0) + Q_n - \chi_n$

(98) $Y_n'(m,\psi_0) = Q_n'(m,\psi_0) + Q_n - \chi_n$

(61) $\Delta N = e^2(\frac{1}{4} - \frac{3}{4}\sin^2\phi)N$

We have the variance of the truncation error:

(99) $\sigma_{\delta N_E'}^2(m,\psi_0) = (\frac{R}{2\gamma})^2 \sum_{n=m+1}^{\infty} Q_n'^2(m,\psi_0)\,\sigma_n^2(\Delta g^0)$

From (Hsu and Zhu, ibid, sec. III) we find:

(100) $|Q_n'(m,\psi_0)| < |Q_n|$ (and $|\bar{Q}_n|$)

Comparing (99) with (68) and (77) respectively, we get

(101) $\sigma_{\delta N_E'}^2(m,\psi_0) < \sigma_{\delta N_E^I}^2(m,\psi_0)$ (and $\sigma_{\delta N_E^{II}}^2(m,\psi_0)$)

It means that the convergence of the series in case of the optimum approximation is better than that of the series in the truncation method of Rapp.

The second kind (using the best squares approximation with a boundary restraint):

In this circumstance, let the coefficients of approximation function $S_m(\psi)$ in the equation (79) be determined by the following minimum criterion (Hsu and Zhu, ibid, sec. V):

(102) $D = \int_{\psi_0}^{\pi} [S(\psi) - S_m(\psi)]^2 \sin\psi\,d\psi\,d\alpha + 2\lambda_1[S(\psi_0) - S_m(\psi_0)]$

$$+ 2\lambda_2\left[\frac{ds(\psi)}{d\psi_0} - \frac{ds_m(\psi)}{d\psi_0}\right] = min$$

in which, λ_1 and λ_2 are undetermined multipliers. Using Lagrange's principle we arrive at approximation coefficients K_n :

(103) $K = \chi^{-1} Q - (\lambda_1 \chi^{-1} U + \lambda_2 \chi^{-1} W)$

(104) $\lambda = A^{-1} B$

where

$$K = [K_1, K_2, \ldots K_m]^T$$

$$\lambda = [\lambda_1, \lambda_2]^T$$

$$Q = [Q_1, Q_2, \ldots Q_m]^T$$

$$A = \begin{bmatrix} U^T \chi^{-1} U, & U^T \chi^{-1} W \\ W^T \chi^{-1} U, & W^T \chi^{-1} W \end{bmatrix} \quad , \quad B = \begin{bmatrix} U^T \chi^{-1} Q \\ W^T \chi^{-1} Q \end{bmatrix}$$

$$U = \begin{bmatrix} \tfrac{1}{2} P_0(\cos\psi_0) \\ \vdots \\ \tfrac{2r+1}{2} P_r(\cos\psi_0) \\ \vdots \\ \tfrac{2m+1}{2} P_m(\cos\psi_0) \end{bmatrix} \quad , \quad W = \begin{bmatrix} 0 \\ \vdots \\ \tfrac{2r+1}{2} r [P_r(\cos\psi_0) - \cos\psi_0 P_r(\cos\psi_0)] \\ \vdots \\ \tfrac{2m+1}{2} m.[P_m(\cos\psi_0) - \cos\psi_0 P_m(\cos\psi_0)] \end{bmatrix}$$

$$\chi = \begin{bmatrix} \chi_{00}, & \chi_{01}, & \cdots & \chi_{0m} \\ \chi_{10}, & \chi_{11}, & \cdots & \chi_{1m} \\ \cdots & \cdots & \cdots & \cdots \\ \chi_{m0}, & \chi_{m1}, & \cdots & \chi_{mm} \end{bmatrix} \quad , \quad \chi_{rs} = \int_0^{\psi_0} P_r(\cos\psi) P_s(\cos\psi) \sin\psi \, d\psi$$

We finally obtain the similar results to (92) and (99) but that the coefficients $K_n(m,\psi_0)$ and $Q'_n(m,\psi_0)$ are now replaced with K_n and Q_n^*, where Q_n^* may be calculated by the following explicit approximate formula (Hsu and Zhu, ibid, eq. (50)):

$$(102) \quad Q_n^* = \frac{1}{n^3(n+1)} \sin\psi_0 P_{n1}(\cos\psi_0) Z(\psi_0) + O\left(\frac{1}{n^4\sqrt{n}}\right)$$

$$(103) \quad Z(\psi_0) = 2 + 9\cos\psi_0 + \frac{1}{\sqrt{2}(1-\cos\psi_0)^{3/2}} - 2S(\psi_0)$$

It should be easy to see that $|Q_n^*| < |Q_n|$. It follows that the convergence of the series in this case is faster than that of the series of Rapp's method.

6. Conclusions

(a) All available procedures to approach Stokes' problem for the ellipsoid as a reference surface could be classified as two groups: the series solution, and the ellipsoidal kernel solution.

(b) The ellipsoidal kernel solution represents a closed iterative form of Stokes' integral. It seems that this kind of solution is convenient for theoretical purposes.

(c) The ellipsoidal kernel method developed by Lelgemann makes the computation of ellipsoidal correction to the geoid undulation by using potential coefficient information possible; however, it can not be directly used for determining ellipsoidal correction to the geoid undulation through combining gravity anomaly information with potential coefficient information.

(d) The method of Rapp, containing the advantage in Lelgemann method and removing its defect, is the best one of all available methods. It can be effectively used for determining highly accurate geoids from potential coefficient information and terrestrial gravity data.

(e) As can be seen from sec. 5, it is better to use the optimum squares approximation to Stokes' function in order to further improve the convergence of solution of the problem.

References

Bjerhammar, A., (1962), On an Explicit Solution of the Gravimetric Boundary Value Problem for an Ellipsoidal Surface of Reference, Rayal Institute of Technology, Stockholm.

Fang Tsün (1965), On the Figure of the Earth's Surface in Consideration of the First Order Term of the Flattening, Institute of Geodesy and Geophysics, Chinese Academy of Sciences, Publication No. 1, Wuchang.

Fang Tsün (1973), Gravimetry and Figure of the Earth, v.2 Scientific Publishing House, Peking.

Hsu Houtze and Zhu Zhuowen (1979), Approximation of Stokes' Function and estimation of Truncation Error, paper presented at XVII General Assembly of the IUGG, Canberra.

Koch, K.R. (1968), Solution of the Geodetic Boundary-Value Problem in Case of a Reference Ellipsoid, Dept. of Geodetic Science Report No. 104, The Ohio State University, Columbus, Ohio.

Lelgemann, D. (1970), On a Solution of Stokes' Problem for the Ellipsoid by a Linear Approximation, Transl. from German.

Molodenskii, M.S., V.F. Eremeev and M.I. Yurkina (1962), Methods for Study of the External Gravitational Field and Figure of the Earth, Transla. from Russian (1960) Jerusalem, Israel Program for Scientific Translations.

Moritz, H. (1980), Advanced Physical Geodesy, Abacus Press, England.

Pellinen, L.P. (1981), Effects of the Earth Ellipticity on Solving Geodetic Boundary Value Problem, Presented to VIII Int'l. Hotine Symposium on Mathematical Geodesy, Como, Italy.

Rapp, R.H. (1981), Ellipsoidal Correction for Geoid Undulation Computations, Dept. of Geodetic Science, Report No. 308, The Ohio State University, Columbus, Ohio.

Zagrebin, D.V. (1956), Dei Theorie des regularisierten Geods. German transl. from Russian (1952), Veroff. Geod. Institut Posdam, Nr. 9, Berlin.

Zagrebin, D.V. (1965), Stokes' Formula for the Case of an Ellipsoidal Level Surface, Transl. from Russian (1941), Linguistic Section, Western Area Branch, Chart Research Division, Aeronautical Chart and Information Center.